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Congruence Properties of the m-ary Partition Function

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If $b(m; n)$ denotes the number of partitions of n into powers of m, then $b(m; m^{r+1}n) \equiv b(m; m^r n) \pmod{\mu^r}$ where $\mu = m$ if m is odd and $\mu = m/2$ if m is even. The existence of such a congruence was conjectured by R. F. Churchhouse and its truth for m a prime was proved by O. Rödseth.

1. Introduction

In Ref. [1], Churchhouse proved a number of results about the binary partition function and also conjectured the following result which was proved in Ref. [2] by Rödseth.

THEOREM. *If $k \geqslant 1$ and $t \equiv 1 \pmod 2$, then*

$$b(2^{2k+2}t) - b(2^{2k}t) \equiv 0 \pmod{2^{3k+2}},$$
$$b(2^{2k+1}t) - b(2^{2k-1}t) \equiv 0 \pmod{2^{3k}}.$$

Furthermore, these congruences hold exactly (i.e., they are false for any higher power of two).

If $b(m; n)$ denotes the number of partitions of n into powers of m, then Churchhouse discussed computer evidence which indicates that some congruence properties similar to the above result must hold for $b(m; n)$.

In this paper, we shall prove

THEOREM 2.

$$b(m; m^{r+1}n) - b(m; m^r n) \equiv 0 \pmod{\mu^r},$$

where $\mu = m$ if m is odd and $\mu = m/2$ if m is even.

Rödseth [2] has proved Theorem 2 when m is a prime.

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2. Generating Functions for $b(m; m^{r+1}n) - b(m; m^r n)$

We shall utilize the following notation.

$$F(m; q) = \sum_{n=0}^{\infty} b(m; n)\, q^n = \prod_{n=0}^{\infty} \frac{1}{(1 - q^{m^n})};$$

$$G(m; q) = (1 - q)\, F(m; q) = F(m; q^m);$$

$$\mathfrak{F}_r(m; q) = \sum_{n=0}^{\infty} (b(m; m^{r+1}n) - b(m; m^r n))\, q^n;$$

$$\rho = \rho_m = e^{2\pi i/m}.$$

Our first object is to study $\mathfrak{F}_r(m; q)$ for $r = 0, 1, 2$, and 3.

$$\begin{aligned}
\mathfrak{F}_0(m; q^m) &= \frac{1}{m} \sum_{j=0}^{m-1} F(m; \rho^j q) - F(m; q^m) \\
&= G(m; q) \left(\frac{1}{m} \left\{ \sum_{j=0}^{m-1} \frac{1}{1 - \rho^j q} \right\} - 1 \right) \\
&= G(m; q) \left(\frac{1}{m} \left\{ \sum_{j=0}^{m-1} \sum_{n=0}^{\infty} (\rho^j q)^n \right\} - 1 \right) \\
&= G(m; q) \left(\sum_{n=0}^{\infty} q^{nm} - 1 \right) \\
&= \frac{G(m; q)\, q^m}{(1 - q^m)}.
\end{aligned}$$

Hence

$$\mathfrak{F}_0(m; q) = \frac{qG(m; q)}{(1 - q)^2}.$$

Now in general

$$\mathfrak{F}_{r+1}(m; q^m) = \frac{1}{m} \sum_{j=0}^{m-1} \mathfrak{F}_r(m; \rho^j q). \tag{2.1}$$

Therefore

$$\mathfrak{F}_1(m; q^m) = G(m; q)\frac{1}{m}\sum_{j=0}^{m-1}\frac{\rho^j q}{(1-\rho^j q)^2}$$

$$= G(m; q)\frac{1}{m}\sum_{j=0}^{m-1}\sum_{n=0}^{\infty} n(\rho^j q)^n$$

$$= G(m; q)\sum_{n=0}^{\infty} mnq^{mn}$$

$$= \frac{mG(m; q)\, q^m}{(1-q^m)^2}.$$

Thus

$$\mathfrak{F}_1(m; q) = \frac{mG(m; q)q}{(1-q)^3}. \tag{2.2}$$

Now by (2.1)

$$\mathfrak{F}_2(m; q^m) = mG(m; q)\cdot\frac{1}{m}\sum_{j=0}^{m-1}\frac{\rho^j q}{(1-\rho^j q)^3}$$

$$= mG(m; q)\cdot\frac{1}{m}\sum_{j=0}^{m-1}\sum_{n=0}^{\infty}\binom{n+1}{2}(\rho^j q)^n$$

$$= mG(m; q)\sum_{n=0}^{\infty}\binom{mn+1}{2} q^{mn}$$

$$= mG(m; q)\sum_{n=0}^{\infty}\left(m^2\binom{n+1}{2} - \binom{m}{2} n\right) q^{mn}$$

$$= \frac{m^3G(m; q)\, q^m}{(1-q^m)^3} - \binom{m}{2}\frac{mG(m; q)\, q^m}{(1-q^m)^2}$$

$$= \frac{m^3G(m; q)\, q^m}{(1-q^m)^3} - \binom{m}{2}\mathfrak{F}_1(m; q^m).$$

Thus

$$\mathfrak{F}_2(m; q) + \binom{m}{2}\mathfrak{F}_1(m; q) = \frac{m^3G(m; q)q}{(1-q)^4}. \tag{2.3}$$

Again by (2.1) applied to (2.3) we find

$$\begin{aligned}
&\mathfrak{F}_3(m; q^m) + \binom{m}{2} \mathfrak{F}_2(m; q^m)\\
&= m^3 G(m; q) \frac{1}{m} \sum_{j=0}^{m-1} \frac{\rho^j q}{(1-\rho^j q)^4}\\
&= m^3 G(m; q) \frac{1}{m} \sum_{j=0}^{m-1} \sum_{n=0}^{\infty} \binom{n+2}{3} (\rho^j q)^n\\
&= m^3 G(m; q) \sum_{n=0}^{\infty} \binom{mn+2}{3} q^{mn}\\
&= m^3 G(m; q) \sum_{n=0}^{\infty} \left(m^3 \binom{n+2}{3} - m \binom{m}{2} n^2 - 2n \binom{m+1}{3}\right) q^{mn}\\
&= \frac{m^6 G(m; q)\, q^m}{(1-q^m)^4} - \frac{2m^4 \binom{m}{2} G(m; q)\, q^m}{(1-q^m)^3} + \frac{m^4 \binom{m}{2} G(m; q)\, q^m}{(1-q^m)^2}\\
&\quad - \frac{2m^3 \binom{m+1}{3} G(m; q)\, q^m}{(1-q^m)^2}\\
&= \frac{m^6 G(m; q)\, q^m}{(1-q^m)^4} - 2m \binom{m}{2}\left(\mathfrak{F}_2(m; q^m) + \binom{m}{2} \mathfrak{F}_1(m; q^m)\right)\\
&\quad + \left(m^3 \binom{m}{2} - 2m^2 \binom{m+1}{3}\right) \mathfrak{F}_1(m; q^m).
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathfrak{F}_3(m; q) + \binom{m}{2} (2m+1)\, \mathfrak{F}_2(m; q) \qquad (2.4)\\
&+ \left(2m \binom{m}{2}^2 - m^3 \binom{m}{2} + 2m^2 \binom{m+1}{3}\right) \mathfrak{F}_1(m; q) = \frac{m^6 G(m; q) q}{(1-q)^5}.
\end{aligned}$$

We are now in a position to prove Theorem 1.

THEOREM 1. *There exist integers $a_i(r) (= a_i(m; r))$ such that*

$$\sum_{i=0}^{r-1} a_i(r)\, \mathfrak{F}_{r-i}(m; q) = \frac{m^{\binom{r+1}{2}} G(m; q) q}{(1-q)^{r+2}}. \qquad (2.5)$$

Furthermore $a_0(r) = 1$, $m \mid a_1(r)$ *if* m *is odd*, $\frac{1}{2}m \mid a_1(r)$ *if* m *is even, and* $m^{2j-2} \mid a_j(r)$ *for* $2 \leqslant j \leqslant r-1$.

Proof. From (2.2), (2.3), and (2.4) we see that our theorem is valid for $r = 1, 2, 3$. We proceed by mathematical induction assuming that Theorem 1 is valid for all integers less than or equal to a particular r (where $r \geqslant 3$). To treat $\mathfrak{F}_{r+1}(m; q)$, we apply (2.1) to (2.5). Thus

$$\sum_{i=0}^{r-1} a_i(r)\, \mathfrak{F}_{r+1-i}(m; q^m) = m^{\binom{r+1}{2}} G(m; q) \frac{1}{m} \sum_{j=0}^{m-1} \frac{\rho^j q}{(1-\rho^j q)^{r+2}}$$

$$= \frac{m^{\binom{r+1}{2}} G(m; q)\, q^m}{(1-q^m)^{r+2}} \cdot P_{r+2}(q),$$

where

$$P_{r+2}(q) = \frac{1}{m} \sum_{j=0}^{m-1} \frac{\rho^j q^{1-m}(1-q^m)^{r+2}}{(1-\rho^j q)^{r+2}}.$$

First we note that $q^m P_{r+2}(q)$ is a polynomial in q since $1 - \rho^j q \mid 1 - q^m$. Furthermore $q^m P_{r+2}(q)$ is a symmetric function of $q, \rho q, \rho^2 q, \ldots, \rho^{m-1} q$, and since the only nonvanishing elementary symmetric functions of these numbers are 1 and $(-1)^{m-1} q^m$, we see that $q^m P_{r+2}(q)$ is a polynomial in q^m with integral coefficients. Since $q^m P_{r+2}(q)$ vanishes when $q = 0$, we thus see that $P_{r+2}(q)$ is itself a polynomial in q^m with integral coefficients. Now the degree of $P_{r+2}(q)$ must be a multiple of m less than or equal to

$$1 - m + (m-1)(r+2) = m(r+1) - r - 1.$$

Thus we may write

$$P_{r+2}(q) = \sum_{h=0}^{r} \Delta_h(r+2)(1-q^m)^h,$$

where the $\Delta_h(r+2)$ are integers,

$$\Delta_0(r+2) = \lim_{q\to 1} \frac{1}{m} \sum_{j=0}^{m-1} \frac{\rho^j q^{1-m}(1-q^m)^{r+2}}{(1-\rho^j q)^{r+2}}$$

$$= \lim_{q\to 1} \frac{1}{m} \frac{q^{1-m}(1-q^m)^{r+2}}{(1-q)^{r+2}}$$

$$= m^{r+1},$$

$$\Delta_1(r+2) = \lim_{q\to 1} \frac{P_{r+2}(q) - m^{r+1}}{1-q^m}.$$

The evaluation of this limit is a simple (but tedious) application of l'Hôpital's rule. The result is

$$\Delta_1(r+2) = m^r(m-r-3).$$

Thus we have

$$\sum_{i=0}^{r-1} a_i(r)\,\mathfrak{F}_{r+1-i}(m; q^m)$$

$$= \frac{m^{\binom{r+2}{2}} G(m; q^m)\, q^m}{(1-q^m)^{r+3}} + m^r(m-r-3) \sum_{i=0}^{r-1} a_i(r)\,\mathfrak{F}_{r-i}(m; q^m)$$

$$+ \sum_{h=2}^{r} \Delta_h(r+2)\, m^{\binom{r+1}{2} - \binom{r-h+2}{2}} \sum_{k=0}^{r-h} a_k(r-h+1)\,\mathfrak{F}_{r+1-h-k}(m; q^m).$$

This last relation defines the $a_i(r+1)$ and from it we may deduce the required properties. First $a_0(r+1) = a_0(r) = 1$,

$$a_1(r+1) = a_1(r) - m^r(m-r-3).$$

Now since $r \geqslant 3$, $m \mid m^r(m-r-3)$. Hence by the induction hypothesis, $m \mid a_1(r+1)$ if m is odd, and $\frac{1}{2}m \mid a_1(r+1)$ if m is even. Now by hypothesis $m^{2j-2} \mid a_j(r)$ for all j, and

$$a_j(r+1) = a_j(r) - m^r(m-r-3)\, a_{j-1}(r)$$

$$+ \sum_{\substack{h+k=j \\ h \geqslant 2 \\ k \geqslant 0}} \Delta_h(r+2)\, m^{\binom{r+1}{2} - \binom{r-h+2}{2}} a_k(r-h+1).$$

On the right side of this equation,

$$m^{2j-2} \mid a_j(r), \quad m^{2j-2} \mid m^{r-2}(m-r-3)\, m^2 a_{j-1}(r).$$

To establish that $m^{2j-2} \mid a_j(r+1)$, we need only show that for $h \geqslant 2$, $k \geqslant 0$, $h+k=j$

$$\binom{r+1}{2} - \binom{r-h+2}{2} + 2k - 2 \geqslant 2j-2.$$

But this is equivalent to proving

$$\binom{h-1}{2} + (r+1)(h-1) - 2h \geqslant 0.$$

In the given domain, the left side of this last inequality is increasing in both variables; hence its minimum value is at $h = 2$, $r = 3$, viz., zero. Hence $m^{2j-2} \mid a_j(r+2)$ for all j.

Thus Theorem 1 is established.

3. Proof of Theorem 2

To establish Theorem 2, we need only show that μ^r divides the coefficients of $\mathfrak{F}_r(m; q)$. By (2.2) this is obvious for $r = 1$. Assume that the theorem holds for all integers less than a given r. Then by Theorem 1

$$\sum_{j=0}^{r-1} a_j(r)\, \mathfrak{F}_{r-j}(m; q) = \frac{m^{\binom{r+1}{2}} G(m; q) q}{(1-q)^{r+2}}.$$

Clearly $\mu^r \mid m^r$ and $m^r \mid m^{\binom{r+1}{2}}$. Thus all the coefficients on the right side are divisible by μ^r. On the left side μ divides $a_1(r)$ and μ^{r-1} divides the coefficients of $\mathfrak{F}_{r-1}(m; q)$, hence μ^r divides the coefficients of $a_1(r)\, \mathfrak{F}_{r-1}(m; q)$. For $j \geqslant 2$, μ^{2j-2} divides $a_j(r)$ and μ^{r-j} divides the coefficients of $\mathfrak{F}_{r-j}(m; q)$. Hence $\mu^r \mid \mu^{r+j-2}$ and μ^{r+j-2} divides the coefficients of $a_j(r)\, \mathfrak{F}_{r-j}(m; q)$. Thus μ^r divides the coefficients of $\mathfrak{F}_r(m; q)$, and Theorem 1 follows by mathematical induction.

References

1. R. F. Churchhouse, Congruence properties of the binary partition function, *Proc. Cambridge Philos. Soc.* **66** (1969), 371–376.
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